Let $\mathcal{C}$ be a class of f.g. R-modules, s.t. $\mathcal{C}$ is closed under

- direct summands (if $M \in \mathcal{C}$, $N \in$ Mod-R and $M \cong N \oplus X$, then $N \in \mathcal{C}$)
- isomorphisms ($M \in \mathcal{C}$, $N \in$ Mod-R w. $N \cong M \Rightarrow N \in \mathcal{C}$)
- direct sums ($M, N \in \mathcal{C} \Rightarrow M \oplus N \in \mathcal{C}$).

Exm: $\mathcal{C} = \{$f.g. R-modules$\}$, $\{$finite length R-modules$\}$, $\{$f.g. proj. R-modules$\}$,

Def: $\mathcal{C}$ has the KRSA property if every module in $\mathcal{C}$ has a unique decomposition into indecomposables.

Cor 8.8: If R is noetherian and End(M) is local for each indec. $M \in \mathcal{C}$, then $\mathcal{C}$ has KRSA [P8.1 + T8.7].

Lemma 8.9 (Fitting) If M_R is a finite-length module and $f \in End(M_R)$, then $M = \ker(f^n) \oplus im(f^n)$ for all suff. large n.

Proof: The chains $M \supseteq im(f) \supseteq im(f^2) \supseteq \ldots$ and

$$0 \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \ldots$$

stabilize, bec. M_R is noetherian and artinian [L2.9].

Let $n \geq 1$ be s.t. $im(f^n) = im(f^{n+1}) = \ldots$, $\ker(f^n) = \ker(f^{n+1}) = \ldots$

Claim: $M = \ker(f^n) \oplus im(f^n)$.

- Let $a \in \ker(f^n) \cap im(f^n) \Rightarrow a = f^n(b)$, $b \in M$

$$0 = f^n(a) = f^{2n}(b) \xRightarrow{\ker f^n = \ker f^{2n}} 0 = f^n(b) = a.$$

- Let $a \in M \Rightarrow f^n(a) = f^{2n}(b)$ for some $b \in M$

$$\Rightarrow f^n(a - f^n(b)) = 0 \Rightarrow a - f^n(b) \in \ker(f^n) \Rightarrow a \in \ker(f^n) + im(f^n) \qquad \square$$

Prop 8.10 If M is an indecomposable module of finite length, then

$E := End(M_R)$ is local w. nil maximal ideal $J(E)$.

Proof: By P8.5 it suffices to show that every $f \in E \setminus E^\times$ is nilpotent.

Let $f \in E \setminus E^\times$, and let $n \geq 1$ be s.t. $M = \ker(f^n) \oplus im(f^n)$ [L8.9]

M indecomposable $\Rightarrow \ker(f^n) = 0$ or $im(f^n) = 0$.

If $\ker(f^n) = 0$, then $im(f^n) = M$, so $f^n \in E^\times$ ↯.

So $im(f^n) = 0 \Rightarrow \ker(f^n) = M \Rightarrow f^n = 0$. □

Cor 8.11 (Krull-Schmidt) If M_R is a finite-length module, then

$M \cong M_1 \oplus \dots \oplus M_n$ with indecomposable M_i. The decomposition is unique up to order and isomorphism and each M_i is strongly indecomposable.

Proof: P8.1 (Existence), and P8.10 + T8.7 (Uniqueness). □

Cor 8.12 The conclusion of C8.11 holds for f.g. modules over right ordinion rings (in particular, over f.d.-algebras)

Proof: F.g. right modules over right ordinion rings have finite length [C3.13]. □

Cor 8.13 Let $0 \neq R$ be right ordinion.

Then R local $\Leftrightarrow$ R has only trivial idempotents.

Proof: „$\Rightarrow$" P8.6

„$\Leftarrow$": R_R has finite length, $End(R_R) \cong R$

By assumption R_R is indecomposable [L8.3] $\overset{P8.10}{\Longrightarrow}$ R local. □

8.3 KRSA in the complete local case

Exm: KRSA fails over comm. noeth. local domains e.g.

$A := \mathbb{C}[x,y,z]/(y^2+x^3-x^2)$

localizing at $(\bar{x},\bar{y},\bar{z})$ gives $R := A_{(\bar{x},\bar{y},\bar{z})}$ local noeth. domain w.

quotient field K, max. ideal $m = (\bar{x},\bar{y},\bar{z})$.

$u := \frac{\bar{y}}{\bar{x}} \in K \setminus R$ satisfies $u^2 = 1-\bar{x} \Rightarrow u$ integral over R

$S := R[u] = R + Ru \cong R[t]/(t^2+\bar{x}-1)$

$S/mS \cong R[t]/(m, t^2+\bar{x}-1) = R/m[t]/(t^2-1) \cong R/m \times R/m$

$\Rightarrow$ S has 2 max. ideals: $M_1 = (m, t-1)$, $M_2 = (m, t+1)$

$M_1 \cap R = M_2 \cap R = m$.

$m \supsetneq (\bar{x},\bar{y}) \supsetneq \underline{0} \Rightarrow ht(m) \geq 2$

$\overset{\text{C.A.}}{\Longrightarrow}$ $ht(M_i) \geq 2 \Rightarrow M_i$ not principal (Krull's Principal Ideal Thm)
Cohen-Seidenberg

$M_1 + M_2 = S \Rightarrow \exists$ SES $\quad 0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to S \to 0$ (S-Mod)

S free $\Rightarrow$ split $\Rightarrow$ $S \oplus (M_1 \cap M_2) \cong M_1 \oplus M_2$ in S-Mod and R-Mod

Each of S, $M_1 \cap M_2$, M_1, M_2 is indecomposable in R-Mod (submodules of K)

Claim: $(M_i)_R \ncong S_R$

If $\varphi: S_R \xrightarrow{\sim} (M_i)_R$, then $\exists \lambda \in K^\times: \varphi(x) = \lambda x \quad \forall x \in S$ (Exc.)

$\Rightarrow S_S \cong (M_i)_S$ ↯ M_i non-principal as S-ideal $\quad \varnothing$

I-adic completion R ring, $I \trianglelefteq R$ (e.g. $R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$, $I = (p)$)

Quotient rings:

$$\cdots \twoheadrightarrow R/I^3 \twoheadrightarrow R/I^2 \twoheadrightarrow R/I \, (\twoheadrightarrow R/R)$$

form an inverse system

Inverse Limit (= colimit):

$$\hat{R} := \hat{R}_I := \varprojlim_n R/I^n := \{ (a_n)_{n\geq 0} \in \bigoplus_{n\geq 0} R/I^n : \forall m \geq n:\ a_m + I = a_n + I \}$$

is a ring (I-adic completion of R)

$j_I : R \to \hat{R}$, $a \mapsto (a)_{n\geq 0}$ is a ring hom.

__Fact__ (Krull's Intersection Theorem) If R is comm. noetherian and $I \subsetneq R$, then $\bigcap_{n\geq 0} I^n = 0$, so $\ker(j_I) = 0$ and $R \hookrightarrow \hat{R}$.

__Def:__ R is I-adically complete if j_I is an isomorphism ($R = \hat{R}$)

__Exm:__ ·) $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ are not p-adically complete, $\hat{\mathbb{Z}}_{p\mathbb{Z}} = \hat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p$

·) $\mathbb{C}[x]$ not (x)-adically complete, completion $\mathbb{C}[[x]]$

__Idea:__ elements of I^n are „small" for large n.

If $\bigcap_{n\geq 0} I^n = 0$, there is a metric d_I on R (fix $0 < c < 1$):

$$d_I(x,y) := \inf \{ c^n : x - y \in I^n \}$$

Then $\hat{R}$ is the completion of R w.r.t. d_I in the sense of Cauchy sequences.

If $M \in \text{Mod-}R$, similarly $\hat{M} := \hat{M}_I := \varprojlim_n M/MI^n$

Def: R ring, $I \trianglelefteq R$. Then idempotents lift modulo I if $\forall$ idempotent $\bar{e} \in R/I$ $\exists$ idempotent $e \in R$ s.t. $e + I = \bar{e}$.

Exm: $\mathbb{Z}/6\mathbb{Z}$ has idempotents $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ but only $\bar{0}, \bar{1}$ lift to idempotents of $\mathbb{Z}$!

Note: If $R = \hat{R}_I$ then $I \subseteq J(R)$

[If $x \in I$, then $1-x$ has inverse $1+x+x^2+\ldots \in \hat{R}$]

Thm 8.14: $I \trianglelefteq R$

(1) If I is nil, idempotents lift modulo I

(2) If $R = \hat{R}_I$, idempotents lift modulo I

Proof: (1) Let $a \in R$ be s.t. $\bar{a} := a + I$ is idempotent, $b := 1 - a$.

$$\Rightarrow ab = ba = a - a^2 \in I \Rightarrow \exists m \geq 1: (ab)^m = 0$$

$$1 = (a+b)^{2m} = \sum_{n=0}^{2m} \binom{2m}{n} a^n b^{2m-n} = \underbrace{\sum_{n=m}^{2m} \binom{2m}{n} a^n b^{2m-n}}_{=: e} + \underbrace{\sum_{n=0}^{m-1} \binom{2m}{n} a^n b^{2m-n}}_{=: f}$$

$$a^m b^m = b^m a^m = 0 \Rightarrow ef = 0$$

$$\Rightarrow \underline{e} = e \cdot 1 = e(e+f) = \underline{e^2}.$$

$$e + I \underset{\substack{\uparrow \\ ab \in I}}{=} a^{2m} + I = a + I.$$

(2) Let $a_1 \in R/I$ be idempotent $(a_0 := 0)$

Inductively: $\forall n \geq 2$ $\exists$ idempotent $a_n \in R/I^n$ s.t. $a_n + I^{n-1} = a_{n-1} \in R/I^{n-1}$

(since $R/I^{n-1} \cong (R/I^n)/(I^{n-1}/I^n)$ and I^{n-1}/I^n is nil, use (1))

Then $(a_n)_{n\geq 0} \in \hat{R}$ works $\square$

<u>Lemma 8.15</u> Let R be a comm. ring, S a module-finite R-algebra.

Then $J(R)S \subseteq J(S)$

<u>Proof:</u> Let M_S be simple. Suffices to show: $M\,J(R)S = 0$.

$MJ(R)$ is an S-submodule of M (since $MJ(R)S \subseteq MSJ(R) \subseteq MJ(R)$)

$\Rightarrow MJ(R) \in \{M, 0\}$,

Since S_R is f.g., also M_R is f.g. $\xRightarrow{\text{Nakayama}}$ $MJ(R)=0$. $\square$

<u>Prop 8.16</u> R comm. noeth. ring, $I \trianglelefteq R$, $R = \hat{R}_I$. If S is a module-finite R-algebra, then S is I-adically complete ($S \cong \varprojlim_n S/SI^n$)

<u>Proof:</u> $j: S \to \hat{S}$, $N := \ker(j) = \bigcap_{n\geq 0} SI^n$

Krull's Intersection Theorem $\Rightarrow N = NI \xRightarrow[\text{Nakayama}]{I \subseteq J(R)} N = 0$.

<u>Surjectivity:</u> Let $S = \langle s_1, \dots, s_m \rangle_R$, $(\bar{a}_n)_{n\geq 0} \in \varprojlim_n S/SI^n$,

$a_n \in S$.

$$\Rightarrow a_{n+1} - a_n = \sum_{j=1}^{m} s_j \lambda_{nj}, \quad \lambda_{nj} \in I^n$$

$$a_1 = \sum_j s_j \gamma_{1j}, \quad \gamma_{1j} \in R$$

$$\Rightarrow a_n = a_1 + (a_2 - a_1) + \dots + (a_n - a_{n-1})$$

$$= \sum_{j=1}^{m} s_j \big(\underbrace{\gamma_{1j} + \lambda_{1j} + \dots + \lambda_{n-1,j}}_{=:\gamma_{n,j}} \big)$$

$\gamma_{n,j} \to \gamma_j$ as $n \to \infty$ (in $\hat{R}$) since $\lambda_{n_j} \in I^n$.

$$a := \sum_{j=1}^{m} s_j \gamma_j \Rightarrow a - a_n = \sum_{j=1}^{m} s_j (\gamma_j - \gamma_{n_j}) \in SI^n. \quad \square$$

<u>Def.</u> A ring R is semilocal if $R/J(R)$ is semisimple

<u>Thm 8.17</u> Let R be a commutative noeth. semilocal ring that is $J(R)$-adically complete. If $0 \neq S$ is a module-finite R-algebra, with only trivial idempotents, then S is local.

<u>Proof:</u> Let $I := J(R)$. By L8.15, $SI \subseteq J(S)$

Consider $S \overset{\varphi}{\twoheadrightarrow} S/SI \overset{\psi}{\twoheadrightarrow} S/J(S)$

R/I semisimple, S/SI module finite over $R/I \to S/SI$ extension

$\Rightarrow J(S/SI) = J(S)/SI$ is nilpotent [T3.6]

$\Rightarrow$ idempotents lift along ψ [T8.14(1)]

By T8.14(2), P8.16 idempotents lift along φ

$\Rightarrow$ idempotents lift modulo $J(S)$.

$S/J(S)$ is semisimple [T3.8]. Since S has only trivial idempotents, $S/J(S)$ has only trivial idempotents $\xrightarrow{\text{Wedderburn-Artin}}$ $S/J(S)$ is a division ring. $\square$

<u>Thm 8.18</u> Let R be a comm. noetherian semilocal ring that is $J(R)$-adically complete. If S is a module-finite R-algebra, then the class of f.g. S-modules has the KRSA property.

(Every f.g. S-module decomposes uniquely into indecomposables)

<u>Proof:</u> By [C8.8] it suffices: if M_S is indecomposable, $E := End(M_S)$ is local. E has only trivial idempotents [L8.3]. Since S_R is f.g., also M_R is f.g., hence $End(M_R)$ is module-finite over R. Since $End(M_S)$ is an R-submodule, E is a module-finite R-algebra. By T8.17, E is local. $\square$